Dealer Intermediation Between Markets

Technical Web Appendix

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Appendix A: Value Functions

Proposition 1.

We derive the linear form of the value functions for each of the three inventory states $s = -1, 0, 1$. For this purpose we conjecture that the optimal standardized B2C quotes $(a(s), b(s)) = (\widehat{a}(s) - x_t, \widehat{b}(s) - x_t)$ are independent from the variable x_t. In proposition 2, we show that this is indeed the case under optimal quote setting. Intuitively, dealers earn a cash flow from intertemporal demand intermediation in the B2C market. The expected cash flow created from the customer relationship should therefore not depend on the price level of the asset under consideration. Hence, the value function cannot depend on the process x_t if the dealer starts from a zero inventory level. We therefore impose the condition $V(0, x_t) = V(0) = V$ for all levels of x_t.

For a positive or negative inventory level, however, the value function generally depends on the level of the asset price because the inventory itself is valuable. Next we determine the functional form of $V(1, x_t)$. The case of $V(-1, x_t)$ is analogous. Recall that the stochastic process x_t has binomial innovations $\Delta x_{t+1} \in \{+\epsilon, -\epsilon\}$ of constant and equal probability $\frac{1}{2}$. To garantee that price setting occurs on the support of the reservation price distribution in $t+1$, we restrict $\epsilon < \overline{\epsilon} = \frac{1}{2d} - \frac{1}{4}S(\overline{\epsilon})$, where $S(\overline{\epsilon})$ denotes the equilibrium B2B spread for the maximal innovations $\overline{\epsilon}$. Note that $S(\overline{\epsilon})$ is a montonically increasing function in $\overline{\epsilon}$.We further assume that dealers earn (pay) interest on the nominal value $rx_t = \frac{1-\beta}{\beta}x_t$ of their positive (negative) inventory. The transition probabilities follow from Assumption 1 as

$$
\begin{array}{lllll}
p_{12} & = & qF^b(R^b - x_{t+1} \leq \widehat{b}(1) - x_{t+1}) & = & q\left(1 + b\left(1\right)d - d\Delta x_{t+1}\right) \\
p_{11} & = & 1 - p_{12} - p_{10} & & \\
p_{10} & = & qF^a\left(R^a - x_{t+1} \geq \widehat{a}(1) - x_{t+1}\right) & = & q\left(1 - a(1)d + d\Delta x_{t+1}\right) \\
p_{01} & = & qF^b(R^b - x_{t+1} \leq \widehat{b}(0) - x_{t+1}) & = & q\left(1 + b\left(0\right)d - d\Delta x_{t+1}\right) \\
p_{00} & = & 1 - p_{01} - p_{0-1} & & \\
p_{0-1} & = & qF^a\left(R^a - x_{t+1} \geq \widehat{a}(0) - x_{t+1}\right) & = & q\left(1 - a(0)d + d\Delta x_{t+1}\right) \\
p_{-10} & = & qF^b(R^b - x_{t+1} \leq \widehat{b}(-1) - x_{t+1}) & = & q\left(1 + b\left(-1\right)d - d\Delta x_{t+1}\right) \\
p_{-1-1} & = & 1 - p_{-10} - p_{-1-2} & = & \\
p_{-1-2} & = & qF^a\left(R^a - x_{t+1} \geq \widehat{a}(-1) - x_{t+1}\right) & = & q\left(1 - a(-1)d + d\Delta x_{t+1}\right)
\end{array}
. \quad (1)
$$

Using the transition probabilities, we express the value functions as

$$
\begin{aligned}
V(1, x_t) \quad = \quad & \frac{1}{2}\beta\left[V(1, x_t + \epsilon)(1 - p_{10}^{+}) + \left[B - b(1)\right]p_{12}^{+} + V(0, x_t + \epsilon)p_{10}^{+} + \left[a(1) + x_t\right]p_{10}^{+}\right] + \\
& + \frac{1}{2}\beta\left[V(1, x_t - \epsilon)(1 - p_{\overline{1}0}^{-}) + \left[B - b(1) - c\right]p_{\overline{1}2}^{-} + V(0, x_t - \epsilon)p_{\overline{1}0}^{-} + \left[a(1) + x_t\right]p_{\overline{1}0}^{-}\right] \\
& + \beta r x_t,
\end{aligned} \quad (2)
$$

where $p_{s_1s_2}^{+}$ and $p_{s_1s_2}^{-}$ denotes the transition probability from inventory state s_1 to s_2 for innovations $\Delta x_{t+1} = +\epsilon$ and $\Delta x_{t+1} = -\epsilon$, respectively. Inspection of equation (2) shows that repeated substitution for the terms $V(1, x_t + \epsilon)$ and $V(1, x_t - \epsilon)$ yields a sequence of discounted terms $\beta^i x_t$ (with $i = 1, 2, 3...$) and a sequence of constants $V(0)$,

B, $b(1)$ and $a(1)$ all independent of x_t. A similar consideration follows from the development of

$$\begin{aligned} V(-1,x_t) &= \frac{1}{2}\beta\left[V(-1,x_t+\epsilon)(1-p^+_{-10}) + [a(-1)-A]\,p^+_{-1-2} + V(0,x_t+\epsilon)p^+_{-10} + [b(-1)+x_t]\,p^+_{-10}\right] + \\ &\quad +\frac{1}{2}\beta\left[V(1,x_t-\epsilon)(1-p^-_{-10}) + [a(-1)-A]\,p^-_{-1-2} + V(0,x_t-\epsilon)p^-_{-10} + [b(-1)+x_t]\,p^-_{-10}\right] \\ &\quad -\beta r x_t \end{aligned}$$

Again sequential substitution gives discounted terms only in $\beta^i x_t$ (with $i = 1, 2, 3...$) and a sequence of constants. Under the usual transversality condition that this sequence has an upper bound, there exist some constant k_x for which the value function can be expressed as

$$\begin{aligned} V(1,x_t) &= V(1) + k_x x_t \\ V(-1,x_t) &= V(-1) - k_x x_t \end{aligned},$$

for the inventory levels 1 and -1, respectively. Next we show that $k_x = 1$. Using

$$\begin{aligned} &\frac{1}{2}\left[V(1,x_t+\epsilon)(1-p^+_{10}) + V(1,x_t-\epsilon)(1-p^-_{10})\right] \\ =\ &\frac{1}{2}V(1,x_t+\epsilon)(1-q(1+d\epsilon-da(1)) + \frac{1}{2}V(1,x_t-\epsilon)(1-q(1-d\epsilon-da(1) \\ =\ &V(1,x_t)(1-q(1-da(1))) - k_x q d\epsilon^2 = V(1,x_t)(1-\mathcal{E}_t\left(p_{10}\right)) - k_x q d\epsilon^2 \end{aligned}$$

and

$$\frac{1}{2}\left[V(0,x_t+\epsilon)p^+_{10} + V(0,x_t-\epsilon)p^-_{10}\right] = V(0,x_t)q(1-da(1)) = V(0,x_t)p_{10},$$

we rewrite the value function as

$$\begin{aligned} V(1,x_t) &= \beta V(1,x_t)(1-p_{10}) - \beta k_x q d\epsilon^2 + \beta V(0,x_t)p_{10} + \beta\left[B-b(1)\right]p_{12} + \beta\left[a(1)+x_t\right]p_{10} + \beta r x_t \\ &= \beta V(1,0)(1-p_{10}) - \beta k_x q d\epsilon^2 + \beta V(0,0)p_{10} + \beta\left[B-b(1)\right]p_{12} + \beta a(1)p_{10} + \\ &\quad +\beta k_x x_t(1-p_{10}) + \beta x_t p_{10} + (1-\beta)x_t. \end{aligned}$$

A comparison of coefficients with $V(1,x_t) = V(1) + k_x x_t$ implies that $k_x = \beta k_x(1-p_{10}) + \beta p_{10} + 1 - \beta$ or $k_x = 1$. The value function for the inventory $s = 1$ is therefore given by $V(1,x_t) = V(1) + x_t$. An analogous argument applies to the inventory $s = -1$ where we find also find $k_x = 1$. Defining the concavity parameter $\nabla = V(0) - V(1)$ implies the linear form in proposition 1.

Appendix B: Optimal B2C Quotes

Proposition 2.

(i) The dealer value function (equation 1 in the paper) can be expanded as

$$\begin{aligned} V(1,x_t) &= \max_{\{a(s),b(s)\}} \beta\mathcal{E}_t \left[\begin{array}{c} [V(1)+x_{t+1}+B-b(1)]\,p_{12}+[V(1)+x_{t+1}]\,p_{11}+ \\ +[V(0)+a(1)+x_t]\,p_{10}+rx_t \end{array} \right] \\ V(0,x_t) &= \max_{\{a(s),b(s)\}} \beta\mathcal{E}_t \left[\begin{array}{c} [V(1)+x_{t+1}-b(0)-x_t]\,p_{01}+V(0)\,p_{00}+ \\ +[V(-1)-x_{t+1}+a(0)+x_t]\,p_{0-1} \end{array} \right] \\ V(-1,x_t) &= \max_{\{a(s),b(s)\}} \beta\mathcal{E}_t \left[\begin{array}{c} [V(0)-b(-1)-x_t]\,p_{-10}+[V(-1)-x_{t+1}]\,p_{-1-1}+ \\ +[V(-1)-x_{t+1}-A+a(-1)]\,p_{-1-2}-rx_t \end{array} \right]. \end{aligned} \tag{3}$$

For each of the three state variables, we find the first order conditions by differentiation with respect to the corresponding quoted B2C prices $a(s)$ and $b(s)$. This implies the 6 first order conditions stated in proposition 2. The second order conditions are trivially fulfilled since the Hessian matrix is $-2d\mathbf{I}_3$ and therefore negative definite.

(ii) It is more difficult to derive the condition on the concavity parameter ∇ which depends on the B2B spread S. From proposition 1, we know that the value function has a linear representation in the state variable x_t. In order to solve for ∇, we can write the value function (3) for optimal B2C quotes as

$$\mathbf{V}(s,x_t) = \beta\mathcal{E}_t\left[\mathbf{M}\mathbf{V}(s,x_{t+1})+\widetilde{\mathbf{\Lambda}}\right] = \beta\mathbf{M}\mathbf{V}(s,x_t)+\mathbf{\Lambda}_0+\mathbf{\Lambda}_x x_t+\mathbf{\Phi} \tag{4}$$

where $\mathbf{M}$ denotes the transition matrix and where we define vectors

$$\mathbf{\Lambda}_0 = \beta \left[\begin{array}{c} \left[-\frac{S}{2}-b(1)\right]p_{12}+a(1)p_{10} \\ -b(0)p_{01}+a(0)p_{0-1} \\ -b(-1)p_{-10}+\left[a(-1)-\frac{S}{2}\right]p_{-1-2} \end{array} \right], \tag{5}$$

$$\mathbf{\Lambda}_x = \beta \left[\begin{array}{c} 1+r \\ p_{0-1}-p_{01} \\ -(1+r) \end{array} \right] = \left[\begin{array}{c} 1 \\ 0 \\ -1 \end{array} \right],$$

$$\mathbf{\Phi} = \beta\mathcal{E}_t \left[\begin{array}{c} \Delta x_{t+1}(p_{12}+p_{11}) \\ \Delta x_{t+1}(p_{01}-p_{0-1}) \\ -\Delta x_{t+1}(p_{-1-1}+p_{-1-2}) \end{array} \right] = \beta \left[\begin{array}{c} -qd\mathcal{E}_t(\Delta x_{t+1})^2 \\ -2qd\mathcal{E}_t(\Delta x_{t+1})^2 \\ -qd\mathcal{E}_t(\Delta x_{t+1})^2 \end{array} \right] = \beta q d\epsilon^2 \left[\begin{array}{c} -1 \\ -2 \\ -1 \end{array} \right].$$

Subtracting the vector $\mathbf{\Lambda}_x x_t$ from both sides in equation (4) we obtain

$$\mathbf{V}(s,0) = \mathbf{V}(s) = \beta\mathbf{M}\mathbf{V}(s,0)+\mathbf{\Lambda}_0+\mathbf{\Phi}.$$

Hence, the concavity parameter $\nabla = V(0)-V(1)$ is implicitly characterized by

$$\mathbf{V}(s)=\begin{bmatrix} V(1)\\ V(0)\\ V(-1)\end{bmatrix}=\begin{bmatrix} V-\nabla\\ V\\ V-\nabla\end{bmatrix}=(\mathbf{I}-\beta\mathbf{M})^{-1}(\Lambda_0+\mathbf{\Phi}). \tag{6}$$

The vector Λ_0 denotes the expected payoffs in each state. It is independent of both the current price process x_t and its innovation Δx_{t+1}. The vector $\mathbf{\Phi}$ captures the state specific adverse selection risk with respect to shocks to the price process x_t. The matrix $\mathbf{M}$ of transition probabilities can be written as

$$\begin{aligned}\mathbf{M} &= \mathcal{E}_t\begin{bmatrix} p_{12}+p_{11} & p_{10} & 0\\ p_{01} & p_{00} & p_{0-1}\\ 0 & p_{-10} & p_{-1-1}+p_{-1-2}\end{bmatrix}=\\ &= \begin{bmatrix} 1-[q\{1-a(1)d\}] & q\{1-a(1)d\} & 0\\ q\{1+b(0)\,d\} & 1-q\{1+b(0)\,d\}-q\{1-a(0)d\} & q\{1-a(0)d\}\\ 0 & q\{1+b(-1)\,d\} & 1-q\{1+b(-1)\,d\}\end{bmatrix}.\end{aligned} \tag{7}$$

We can then rewrite

$$\begin{aligned}\mathbf{M} &= \mathbf{I}+\frac{1}{\beta}\mathbf{M}^*=\\ &= \mathbf{I}+\frac{1}{\beta}\begin{bmatrix} -[q\beta\{1-a(1)d\}] & q\beta\{1-a(1)d\} & 0\\ q\beta\{1+b(0)\,d\} & -q\beta\{1+b(0)\,d\}-q\beta\{1-a(0)d\} & q\beta\{1-a(0)d\}\\ 0 & q\beta\{1+b(-1)\,d\} & -q\beta\{1+b(-1)\,d\}\end{bmatrix}.\end{aligned}$$

and substition into

$$(\mathbf{I}-\beta\mathbf{M})\,\mathbf{V}(s)-\Lambda_0-\mathbf{\Phi}=\mathbf{0}$$

implies

$$[(1-\beta)\mathbf{I}-\mathbf{M}^*]\,\mathbf{V}(s)-\Lambda_0-\mathbf{\Phi}=\mathbf{0}$$

Note that in the case that $\beta\to 1$, we can divide the how system by $q\beta$ and the rate q becomes irrelavant for the determination of equilibrium schedule.

Substituting the relevant elements of (1) into (5) and using (**??**), we can rewrite

$$(\mathbf{I}-\beta\mathbf{M})\,\mathbf{V}(s)-\Lambda_0-\mathbf{\Phi}=\mathbf{0}$$

or

$$\begin{bmatrix} 8\beta q+\beta d^2q\left(4\nabla^2+S^2-16\epsilon^2\right)+4d\left\{2\nabla\left(2+\beta\left(q-2\right)\right)-\beta qS+4V(0)\left(\beta-1\right)\right)\}\\ V(0)-\dfrac{\beta q\left\{4d^2\epsilon^2-\left(d\nabla-1\right)^2\right\}}{2d\left(\beta-1\right)}\\ 8\beta q+\beta d^2q\left(4\nabla^2+S^2-16\epsilon^2\right)+4d\left\{2\nabla\left(2+\beta\left(q-2\right)\right)-\beta qS+4V(0)\left(\beta-1\right)\right)\}\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

The second equation can be solved for $V(0)$ in terms of ∇. The first and third equations are identical and we substitute $V(0)$ into either to obtain

$$f_{b2c}\left(\nabla, S, \epsilon^2, q, d\right) = \frac{1}{4}\left(\beta dq\nabla^2\right) + \nabla\left(-1+\beta-\frac{3\beta q}{2}\right) - \frac{\beta q\left\{S\left(dS-4\right)+16d\epsilon^2\right\}}{16} = 0. \tag{8}$$

This B2C schedule characterizes the inventory concavity parameter ∇ of a dealer's value function under optimal B2C quotes and for any B2B spread S. It is depicted in Figure 2.

Appendix C: Competitive Pricing in the B2B Market

Proposition 3.

From assumption 3, we denote by $(n(1), n(0), n(-1)) > 0$ the number of traders with inventories $1, 0$, and -1, respectively. Liquidity at the best B2B ask price is only demanded by dealers who experience an negative inventory shock from -1 to -2 and are therefore forced to rebalance. The respective probability p_{-1-2} (see equations (1)) is given by $q\left(1-a(-1)d+d\epsilon\right)$ when $\Delta x_{t+1} = \epsilon$ (with probability $\frac{1}{2}$) and $q\left(1-a(-1)d-d\epsilon\right)$ when $\Delta x_{t+1} = -\epsilon$ (with probability $\frac{1}{2}$). The liquidity supplying dealer (at the ask) experiences an expected loss if the liquidity demand is more likely to occur for $\Delta x_{t+1} = \epsilon$ than $\Delta x_{t+1} = -\epsilon$. If market orders (due to rebalancing needs) in the B2B market were unrelated to the dynamics of Δx_{t+1}, then the expected (adverse selection) loss L^A of liquidity provision at the ask would follow as

$$L^A = \frac{1}{2}\epsilon + \frac{1}{2}(-\epsilon) = 0.$$

But since execution probabilities for limit order supplies depend on Δx_{t+1}, we have instead

$$L^A = \text{prob}(\Delta x_{t+1} = \epsilon \mid \text{Execution})\ \epsilon + \text{prob}(\Delta x_{t+1} = -\epsilon \mid \text{Execution})\ (-\epsilon), \tag{9}$$

where $\text{prob}(\Delta x_{t+1} = \epsilon\ |\text{Execution}) > \frac{1}{2}$ denotes the probability of $\Delta x_{t+1} = \epsilon$ conditional on execution of the liquidity supply at the ask. Using Bayes rule implies

$$\text{prob}(\Delta x_{t+1} \ = \ \epsilon \mid \text{Execution}) = \frac{\text{prob}(\Delta x_{t+1} = \epsilon \cap \text{Execution})}{\text{prob}(\Delta x_{t+1} = \epsilon \cap \text{Execution}) + \text{prob}(\Delta x_{t+1} = -\epsilon \cap \text{Execution})} \tag{10}$$

$$\text{prob}(\Delta x_{t+1} \ = \ -\epsilon \mid \text{Execution}) = \frac{\text{prob}(\Delta x_{t+1} = -\epsilon \cap \text{Execution})}{\text{prob}(\Delta x_{t+1} = \epsilon \cap \text{Execution}) + \text{prob}(\Delta x_{t+1} = -\epsilon \cap \text{Execution})}. \tag{11}$$

We calculate the expected number of (unit) market order as $n(-1)q\left(1-a(-1)d \pm d\epsilon\right)$ (for $\Delta x_{t+1} = \pm\epsilon$, respectively) and the number of (unit) liquidity supplies at the best ask as $n(1)$. The execution probability for each liquidity supplying dealer then follows as

$$\text{prob}(\Delta x_{t+1} \ = \ \epsilon \cap \text{Execution}) = \frac{1}{2}\frac{n(-1)q\left(1-a(-1)d+d\epsilon\right)}{n(1)}$$

$$\text{prob}(\Delta x_{t+1} \ = \ -\epsilon \cap \text{Execution}) = \frac{1}{2}\frac{n(-1)q\left(1-a(-1)d-d\epsilon\right)}{n(1)}.$$

Both expressions are bounded between 0 and 1 for $\frac{1}{2}q < \frac{n(1)}{n(-1)}$. This is secured by assumption 3. Substitution into equations (10) and (11) implies

$$\begin{aligned}
\text{prob}(\Delta x_{t+1} &= \epsilon \mid \text{Execution}) = \frac{(1-a(-1)d+d\epsilon)}{2(1-a(-1)d)} = \frac{1}{2} + \frac{d\epsilon}{2(1-a(-1)d)} \\
\text{prob}(\Delta x_{t+1} &= -\epsilon \mid \text{Execution}) = \frac{(1-a(-1)d-d\epsilon)}{2(1-a(-1)d)} = \frac{1}{2} - \frac{d\epsilon}{2(1-a(-1)d)}.
\end{aligned}$$

The expected loss of B2B liquidity supply at the best ask stated in (9) follows as

$$L^A = \frac{d\epsilon^2}{[1-a(-1)d]} = \frac{\epsilon^2}{\frac{1}{d} - a(-1)} = \frac{\epsilon^2}{\frac{1}{d} - \frac{S}{4} - \frac{1}{2d}} = \frac{2\epsilon^2}{\frac{1}{d} - \frac{S}{2}},$$

and an analogous expression holds for $L^B = L^A = L$.

The equilibrium condition equalizes the adverse selection costs L^A with benefits of a balanced inventory ∇, the transaction revenue $\frac{S}{2}$ and order processing costs τ. If all rents from liquidity disappear under perfect supply, we obtain as the B2B equilibrium condition

$$f_{b2b}\left(\nabla, S, \epsilon^2, q, d\right) = \tau - \frac{S}{2} - \nabla + \frac{4d\epsilon^2}{2 - Sd} = 0. \tag{12}$$

and (for $S \geq 0$)

$$\begin{aligned}
A &= \max(L - \nabla + \tau, 0) &&= \tfrac{S}{2} \\
B &= \min(-L + \nabla - \tau, 0) &&= -\tfrac{S}{2}
\end{aligned}.$$

Appendix D: Existence and Uniqueness of the Equilibrium

Proposition 4:

First, we show that the two equilibrium schedules (8) and (12) have exactly two intersections in the $(\frac{S}{2}, \nabla)$ space as long as the volatility ϵ^2 of the midprice process x_t is below some treshhold $\bar{\epsilon}^2$. This situation is graphed in Figure 2. Second, we argue that only one of the two equilibria is stable. Third, for high levels of volatility with $\epsilon^2 > \bar{\epsilon}^2$ no equilibrium exists in which both the B2B and B2C market function simultaneously.

To characterize the shape of the B2C equilibrium schedule, we calculate the partial derivatives of the implicit function f_{b2c} giving

$$\begin{bmatrix} \frac{\partial f_{b2c}}{\partial S} \\ \frac{\partial f_{b2c}}{\partial \nabla} \\ \frac{\partial f_{b2c}}{\partial \epsilon^2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{8}\beta q\,(dS-2) > 0 \\ -1 + \frac{1}{2}\beta\left[2\,(1-q) + qd\left(\nabla - \frac{1}{d}\right)\right] < 0 \\ -\beta dq < 0 \end{bmatrix}. \tag{13}$$

We have $\frac{\partial f_{b2c}}{\partial S} > 0$ because the uniform distribution was restricted to have $\frac{S}{2} < \frac{1}{d}$. Moreover, $\frac{\partial f_{b2c}}{\partial \nabla} < 0$, because $q < 1$ and $\nabla < \frac{1}{d}$. To verify the condition $\nabla < \frac{1}{d}$, take into consideration that the ask quote $a(1) = \frac{1}{2}\left(\frac{1}{d} - \nabla\right) > 0$ in equation (3) in the paper needs to be positive. The B2C schedule has the derivatives

$$\frac{\partial \nabla_{b2c}}{\partial S} = -\frac{\frac{\partial f_{b2c}}{\partial S}}{\frac{\partial f_{b2c}}{\partial \nabla}} = \frac{\frac{1}{8}\beta q\,(dS-2)}{-1 + \frac{1}{2}\beta\left[2\,(1-q) + qd\left(\nabla - \frac{1}{d}\right)\right]} > 0 \quad \text{and} \quad \frac{\partial^2 \nabla_{b2c}}{\partial S^2} < 0.$$

In the $(\frac{S}{2}, \nabla)$ space the B2C schedule is therefore increasing in S with a decreasing slope.

Next, we examine the B2B schedule (12). Its intercept with the vertical axis is found by evaluating equation (12) at $S = 0$, which gives $2d\epsilon^2 + \tau$. The B2B schedule has derivatives

$$\frac{\partial \nabla_{b2b}}{\partial S} = -\frac{1}{2} + \frac{4d^2\epsilon^2}{(2 - Sd)^2} \quad \text{and} \quad \frac{\partial^2 \nabla_{b2b}}{\partial S^2} < 0. \tag{14}$$

At $S = 0$, we find $\frac{\partial \nabla_{b2b}}{\partial S} = -\frac{1}{2} + 2d^2\epsilon^2 < 0$, because the maximum value of ϵ^2 is $\frac{1}{4d^2}$. Equation (12) is quadratic. Its minimum is obtained for $\frac{S}{2} = \frac{1}{d} - \sqrt{2\epsilon^2}$. For $\frac{1}{d} - \sqrt{2\epsilon^2} < \frac{S}{2} < \frac{1}{d}$, the slope is positive. Importantly, $\frac{\partial^2 \nabla_{b2b}}{\partial S^2} > 0$ for the B2B schedule and $\frac{\partial^2 \nabla_{b2c}}{\partial S^2} < 0$ for the B2C schedule implies that both schedules intersect exactly twice as long as the volatility ϵ^2 is not too large. Of the two equilibria Z_L and Z_H shown in Figure 2, only Z_L with lower values of S and ∇ is stable. Deviation of a liquidity supplier in the B2B market to a lower spread S immediately attracts all the market orders from other dealers. The less favorable B2B quotes become irrelevant. The reverse argument does not hold, which demonstrates the stability of equilibrium Z_L. Finally, as ϵ^2 becomes large, the B2B and B2C schedule no longer intersect and no market equilibrium exists. The volatility level ϵ^2 at which both schedule touch in one tangency point characterizes the threshold value $\bar{\epsilon}^2$ for breakdown of the joint equilibrium in both markets.

Appendix E: Welfare

Consumer Surplus without Intermediation

In this case only matching transaction demands at the bid and the ask are transacted at the common value price x_t. Here we to determine $E(Tr)$. Note that $y_t(i)$ is a binomial variable of arrival of a customer (i) in a particular time period:

$$y(i) = \begin{cases} 1 & \text{with prob.} \quad q \\ 0 & \text{with prob.} \quad 1 - q \end{cases}$$

Clearly $E[y(i)] = q$ and $Var[y(i)] = q(1 - q)$. Next consider the distribution of

$$Y = \sum_{i=1}^{N} y(i).$$

where we now assume that The sum of N independent binomial variables converges to a normal distribution for large N. We can state the moments as $\mu_Y = E[Y] = Nq$ and $\sigma_Y^2 = Var(Y) = \sum_{i=1}^{N} Var[y(i)] = Nq(1 - q)$. For two normally distributed random variables Y^A and Y^B, we next determine the distribution of

$$Tr = Min(Y^A, Y^B).$$

The joint normal distribution of two independent (identically distributed) normal distribution is given by

$$f(Y^A, Y^B) = \frac{1}{2\pi\sigma_Y^2} \exp\left[-\frac{(Y^A - \mu_Y)^2}{2\sigma_Y^2} - \frac{(Y^B - \mu_Y)^2}{2\sigma_Y^2}\right] = f(Y^A)f(Y^B),$$

where $f(Y^A)$ and $f(Y^B)$ are the univariate normal distribution with

$$f(Y = t) = \frac{1}{\sqrt{2\pi}\sigma_Y} \exp\left[-\frac{(t - \mu_Y)^2}{2\sigma_Y^2}\right]$$

with a cumulative distribution given by $\Phi(Y)$. Now call Y^A the smaller of the two random variables. Conditional on Y^A, we can determine the probability $f(Y^B \geq Y^A \mid Y^A)$ for $Y^B \geq Y^A$. Then

$$f(Tr = t) = f(Y^A = t) f(Y^B \geq Y^A \mid Y^A = t).$$

To see this, note that $f(Tr = t) = f\left(Y^A = t, Y^B \geq Y^A\right)$. The previous result then follows from the relationship between joint, conditional and marginal densities. Next we determine

$$\begin{aligned} f(Y^B &\geq Y^A \mid Y^A = t) = \left[f(Y^A = t)^{-1}\right] \frac{1}{2\pi\sigma_Y^2} \int_{Y^B=t}^{\infty} \exp\left[-\frac{(t-\mu_Y)^2}{2\sigma_Y^2} - \frac{(Y^B-\mu_Y)^2}{2\sigma_Y^2}\right] dY^B \\ &= \left[f(Y^A = t)^{-1}\right] \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\left[-\frac{(t-\mu_Y)^2}{2\sigma_Y^2}\right] \frac{1}{\sqrt{2\pi\sigma_Y^2}} \int_{Y^B=t}^{\infty} \exp\left[-\frac{(Y^B-\mu_Y)^2}{2\sigma_Y^2}\right] dY^B = \\ &= \left[f(Y^A = t)^{-1}\right] \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\left[-\frac{(t-\mu_Y)^2}{2\sigma_Y^2}\right] \left[1 - \Phi(t)\right]. \end{aligned}$$

Since $f(Y^A = t) = \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\left[-\frac{(t-\mu_Y)^2}{2\sigma_Y^2}\right]$,we have

$$f(Y^B \geq Y^A \mid Y^A = t) = 1 - \Phi(t)$$

and

$$f(Tr = t) = \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\left[-\frac{(t-\mu_Y)^2}{2\sigma_Y^2}\right] \left[1 - \Phi(t)\right].$$

The next step is to determine the expected value $E(Tr)$.

$$\begin{aligned} E(Tr) &= \int_{-\infty}^{\infty} t \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\left[-\frac{(t-\mu_Y)^2}{2\sigma_Y^2}\right] \left[1 - \Phi(t)\right] dt \\ &= \int_{-\infty}^{\infty} t \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\left[-\frac{(t-\mu_Y)^2}{2\sigma_Y^2}\right] - \int_{-\infty}^{\infty} t \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\left[-\frac{(t-\mu_Y)^2}{2\sigma_Y^2}\right] \Phi(t) dt \\ &= \mu_Y - \int_{-\infty}^{\infty} t \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\left[-\frac{(t-\mu_Y)^2}{2\sigma_Y^2}\right] \Phi(t) dt \\ &= \mu_Y - \int_{-\infty}^{\infty} t \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\left[-\frac{(t-\mu_Y)^2}{2\sigma_Y^2}\right] \left\{ \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\left[-\frac{(t-\mu_Y)^2}{2\sigma_Y^2}\right] dt \right\} dt. \end{aligned}$$

We can approximate the cumulative distribution around $t = \mu_Y$ using a first order Taylor's approximation, which yields

$$\int_{-\infty}^{t} \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\left[-\frac{(t-\mu_Y)^2}{2\sigma_Y^2}\right] dt \approx \frac{1}{2} + \frac{t-\mu_Y}{\sqrt{2\pi\sigma_Y^2}}.$$

The first term is $\frac{1}{2}$because the normal distribution is symmetric. The second term is obtained from

$$\frac{d\left(\int_{-\infty}^{t} \exp\left[-\frac{(t-\mu_Y)^2}{2\sigma_Y^2}\right] dt\right)}{dt}|_{t=\mu_Y} = \exp\left[-\frac{(t-\mu_Y)^2}{2\sigma_Y^2}\right] |_{t=\mu_Y} = 1.$$

This allows us to simplifies the integral to

$$E(Tr) \approx \mu_Y - \int_{-\infty}^{\infty} t \left[\frac{1}{2} + \frac{t-\mu_Y}{\sqrt{2\pi\sigma_Y^2}}\right] \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\left[-\frac{(t-\mu_Y)^2}{2\sigma_Y^2}\right] dt.$$

We can define a variable substitution with $u = t - \mu_Y$, thus

$$\begin{aligned}
E(Tr) &\approx \mu_Y - \int_{-\infty}^{\infty} (u+\mu_Y)\left[\frac{1}{2} + \frac{u}{\sqrt{2\pi\sigma_Y^2}}\right]\frac{1}{\sqrt{2\pi\sigma_Y^2}}\exp\left[-\frac{u^2}{2\sigma_Y^2}\right]du = \\
&= \mu_Y - \frac{1}{\sqrt{2\pi\sigma_Y^2}}\int_{-\infty}^{\infty}\left[\frac{1}{2}\mu_Y\right]\exp\left[-\frac{u^2}{2\sigma_Y^2}\right]du - \frac{1}{\sqrt{2\pi\sigma_Y^2}}\int_{-\infty}^{\infty}\left[\frac{1}{2} + \frac{\mu_Y}{\sqrt{2\pi\sigma_Y^2}}\right]u\exp\left[-\frac{u^2}{2\sigma_Y^2}\right]du \\
&\quad - \frac{1}{\sqrt{2\pi\sigma_Y^2}}\int_{-\infty}^{\infty}\frac{u^2}{\sqrt{2\pi\sigma_Y^2}}\exp\left[-\frac{u^2}{2\sigma_Y^2}\right]du \\
&= \mu_Y - A_3 - A_2 - A_1
\end{aligned}$$

as

$$(u+\mu_Y)\left[\frac{1}{2} + \frac{u}{\sqrt{2\pi\sigma_Y^2}}\right] = -\frac{1}{2}\mu_Y + \left[\frac{1}{2} + \frac{\mu_Y}{\sqrt{2\pi\sigma_Y^2}}\right]u + \frac{u^2}{\sqrt{2\pi\sigma_Y^2}}.$$

For the three expressions A_1, A_2 and A_3 we obtain

$$\begin{aligned}
A_1 &= \frac{1}{\sqrt{2\pi\sigma_Y^2}}\frac{1}{\sqrt{2\pi\sigma_Y^2}}\int_{-\infty}^{\infty}u^2\exp\left[-\frac{u^2}{2\sigma_Y^2}\right]du = \frac{1}{\sqrt{2\pi\sigma_Y^2}}\sigma_Y^2 = \frac{\sigma_Y}{\sqrt{2\pi}} \\
A_2 &= \frac{1}{\sqrt{2\pi\sigma_Y^2}}\int_{-\infty}^{\infty}\left[\frac{1}{2} + \frac{\mu_Y}{\sqrt{2\pi\sigma_Y^2}}\right]u\exp\left[-\frac{u^2}{2\sigma_Y^2}\right]dt = \frac{1}{\sqrt{2\pi\sigma_Y^2}}\left[\frac{1}{2} + \frac{\mu_Y}{\sqrt{2\pi\sigma_Y^2}}\right]\int_{-\infty}^{\infty}u\exp\left[-\frac{u^2}{2\sigma_Y^2}\right]dt = 0 \\
A_3 &= \frac{1}{\sqrt{2\pi\sigma_Y^2}}\int_{-\infty}^{\infty}\frac{1}{2}\mu_Y\exp\left[-\frac{u^2}{2\sigma_Y^2}\right]dt = \frac{1}{2}\mu_Y\frac{1}{\sqrt{2\pi\sigma_Y^2}}\int_{-\infty}^{\infty}\exp\left[-\frac{u^2}{2\sigma_Y^2}\right]dt = \frac{1}{2}\mu_Y.
\end{aligned}$$

The expression for A_1 was obtained from: $\int_0^{\infty} u^2\exp\left[-\alpha^2u^2\right]du = \frac{\sqrt{\pi}}{4\alpha^3}$ (Zeidler (2004), Page 187 equation (4)). For A_3, we appeal to $\int_0^{\infty}\exp\left[-\alpha^2u^2\right]dt = \frac{\sqrt{\pi}}{2\alpha}$. (Zeidler (2004), Page 187 equation (3)) We obtain

$$E(Tr) \approx \mu_S - A_1 - A_2 - A_3 = \mu_Y - \frac{\sigma_Y}{\sqrt{2\pi}} - 0 - \frac{1}{2}\mu_Y = \frac{1}{2}\mu_Y - \frac{1}{\sqrt{2\pi}}\sigma_Y.$$

Using $\mu_Y = E[Y] = Nq$ and $\sigma_Y^2 = Var(Y) = \sum_{i=1}^{N} Var[y(i)] = Nq(1-q)$, we get

$$E(Tr) \approx \frac{1}{2}\mu_Y - \frac{1}{\sqrt{2\pi}}\sigma_Y = \frac{1}{2}Nq - \frac{1}{\sqrt{2\pi}}\sqrt{Nq(1-q)} = Nq\left[\frac{1}{2} - \frac{1}{\sqrt{2\pi}}\sqrt{\frac{(1-q)}{Nq}}\right] < \frac{1}{2}Nq = \frac{1}{2}E[Y].$$

It follows that the expected number of trades is lower than the half the expected rate of customer arrival and the expected customer welfare for the case of spot matching (no dealer intermediation) follows as

$$W^{Spot} = \frac{1}{d}E(Tr) = \frac{Nq}{2d}\left[1 - \frac{2}{\sqrt{2\pi}}\sqrt{\frac{(1-q)}{Nq}}\right].$$

Customer Welfare without Intermediation

Next we determine the consumer welfare under intermediation. As can be seen from Proposition 2, the baseline price mark-up due to monopolistic pricing is given by $\frac{1}{2d}$. If all inventory states are roughly equally likely to occur, it implies that the average price mark-up is given by

$$Av\_Markup = \frac{1}{2d} + \frac{1}{3}\frac{S}{4}.$$

The probability of a transaction between customer and the dealer in each time period is then given by:

$$P\left[y(i)=1\right] = q\left[\frac{1}{d} - Av\_Markup\right]d = q\left[\frac{1}{d} - \frac{1}{2d} - \frac{1}{3}\frac{S}{4}\right]d = q\left[\frac{1}{2} - \frac{1}{3}\frac{S}{4}d\right]$$

The expression for the above probability $P\left[y(i)=1\right]$ is obtained as follows: d is the density of the uniform distribution from which reservation prices are drawn; the maximum markup is $\frac{1}{d}$ so $\frac{1}{d} - Av\_Markup$ is the range of reservation prices over which a quote will be accepted and q is the probability that a customer requests a quote in each period. Note that the average transaction surplus without intermediation was given by $\frac{1}{d} = \left[\frac{1}{d} - (\text{Markup} \equiv 0)\right]$. In the case of intermediation, it is the difference between the highest valuation $\frac{1}{d}$ and the average mark-up $\frac{1}{2d} + \frac{1}{3}\frac{S}{4}$.[1] The expected transaction surplus per transaction follows as

$$\frac{1}{d} - \left(\frac{1}{2d} + \frac{1}{3}\frac{S}{4}\right) = \frac{1}{2d} - \frac{1}{3}\frac{S}{4}$$

and for N customers we obtain the expression for the total customer surplus, which follows as the product of the number of customers, the probability of a transaction for each customer and the average surplus per transaction; hence

$$W^{Dealer} = N\ P\left[y(i)=1\right]\left[\frac{1}{2d} - \frac{1}{3}\frac{S}{4}\right] = \frac{qN}{d}\left[\frac{1}{2} - \frac{1}{3}\frac{S}{4}d\right]^2 = \frac{qN}{4d}\left[1 - \frac{Sd}{6}\right]^2$$

Monopolistic dealer intermediation is the preferable to a spot matching if and only if

$$\frac{W^{Spot}}{W^{Dealer}} = \frac{\frac{Nq}{2d}\left[1 - \frac{2}{\sqrt{2\pi}}\sqrt{\frac{(1-q)}{Nq}}\right]}{\frac{qN}{4d}\left[1 - \frac{Sd}{6}\right]^2} = \frac{2\left[1 - \frac{2}{\sqrt{2\pi}}\sqrt{\frac{(1-q)}{Nq}}\right]}{\left[1 - \frac{Sd}{6}\right]^2} < 1.$$

Substituing $\overline{\mu} = Nq$ allows us to rewrite this condition as

$$\mu_Y < \overline{\mu} = \frac{2\,(1-q)}{\pi}\left[\frac{(72)^2}{d^2S^2 - 12dS - 36}\right]$$

It is easy to show both this result and that $d^2S^2 - 12dS - 36$ is increasing in Sd for $\frac{S}{2} < \frac{1}{d}$. A Mathematica file is available on request.

[1] Here we use the assumption that the distribution of reservation prices is uniform

Appendix F: Competition

Assumption

Customers arriving with probability q in every period come in two types as (i) unsophisticated customers with the same reservation price distribution and trade immediacy as before and (ii) sophisticated patient customers who accept trades if the transaction price is $\varepsilon > 0$ worse than the most favorable reservation price of any dealer in the market, that is an ask price of $-\nabla + x_t + \varepsilon$ and a bid price of $+\nabla + x_t - \varepsilon$ with $\varepsilon \to 0$. We assume that the share of unsophisticated and sophisticated traders is $1-\lambda$ and $\lambda < 1$, respectively.

New Equilibrium Conditions

The dealers cannot earn any rents on the sophisticated customers if $\varepsilon \to 0$. They nevertheless influence the inventory dynamics of the dealer. Dealers with a balanced inventory will never make any transaction with the sophisticated customers as their reservation price are not favorable enough. However, dealers with a positive (negative) inventory will undertake sell/ask (buy/bid) transactions with sophisticated customers. This implies that the transition matrix $\mathbf{M}$ will change qualitatively for the two entries p_{10} and p_{-10} by a term $q\lambda$. The new probabilities are

$$\begin{array}{lllll}
p'_{12} & = & qF^b(R^b - x_{t+1} \leq \widehat{b}(1) - x_{t+1}) & = & q'\left(1 + b\left(1\right)d - d\Delta x_{t+1}\right) \\
p'_{11} & = & 1 - p'_{12} - p'_{10} & & \\
p'_{10} & = & qF^a\left(R^a - x_{t+1} \geq \widehat{a}(1) - x_{t+1}\right) + q\lambda & = & q'\left(1 - a(1)d + d\Delta x_{t+1}\right) + q\lambda \\
p'_{01} & = & qF^b(R^b - x_{t+1} \leq \widehat{b}(0) - x_{t+1}) & = & q'\left(1 + b\left(0\right)d - d\Delta x_{t+1}\right) \\
p'_{00} & = & 1 - p'_{01} - p'_{0-1} & & \\
p'_{0-1} & = & qF^a\left(R^a - x_{t+1} \geq \widehat{a}(0) - x_{t+1}\right) & = & q'\left(1 - a(0)d + d\Delta x_{t+1}\right) \\
p'_{-10} & = & qF^b(R^b - x_{t+1} \leq \widehat{b}(-1) - x_{t+1}) + q\lambda & = & q'\left(1 + b\left(-1\right)d - d\Delta x_{t+1}\right) + q\lambda \\
p'_{-1-1} & = & 1 - p'_{-10} - p'_{-1-2} & = & \\
p'_{-1-2} & = & qF^a\left(R^a - x_{t+1} \geq \widehat{a}(-1) - x_{t+1}\right) & = & q'\left(1 - a(-1)d + d\Delta x_{t+1}\right)
\end{array}, \tag{15}$$

where $q' = q(1-\lambda)$ represents the new arrival probability for an unsophisticated customer.

The dealers now discriminate between sophisticated and unsophisticated customers. The first order conditions in Proposition 2 for the optimal price quotes (in terms of ∇, d and S) for the unsophisticated customers remain unchanged. However, the concavity parameter ∇ itself changes.

The dealer value maximization problem changes to

$$V(1,x_t) = \max_{\{a(s),b(s)\}} \beta\mathcal{E}_t \left[\begin{array}{c} [V(1)+x_{t+1}+B-b(1)]\,p'_{12}+[V(1)+x_{t+1}]\,p'_{11}+ \\ +[V(0)+a(1)+x_t]\,(p'_{10}-q\lambda)+[V(0)-\nabla+x_t]\,q\lambda+rx_t \end{array} \right] \tag{16}$$

$$V(0,x_t) = \max_{\{a(s),b(s)\}} \beta\mathcal{E}_t \left[\begin{array}{c} [V(1)+x_{t+1}-b(0)-x_t]\,p'_{01}+V(0)\,p'_{00}+ \\ +[V(-1)-x_{t+1}+a(0)+x_t]\,p'_{0-1} \end{array} \right]$$

$$V(-1,x_t) = \max_{\{a(s),b(s)\}} \beta\mathcal{E}_t \left[\begin{array}{c} [V(0)-b(-1)-x_t]\,(p'_{-10}-q\lambda)+[V(0)-\nabla-x_t]\,q\lambda+ \\ [V(-1)-x_{t+1}]\,p'_{-1-1}+[V(-1)-x_{t+1}-A+a(-1)]\,p'_{-1-2}-rx_t \end{array} \right],$$

where the new term in the first equation $p'_{10}-q\lambda = q'\,(1-a(1)d+d\Delta x_{t+1})$ represents the diminished likelihood of moving to the balanced inventory state by making a monopolistic profit $a(1)+x_t$ as opposed to a rebalancing at the lower reservation price $-\nabla+x_t$ to the new state $V(0)$ which generates a new value $V(0)-\nabla+x_t$ (instead of $[V(0)+a(1)+x_t]$) with probability $q\lambda$. This implies a new expected loss of $-q\lambda\nabla$ in state $s=1$ and a similar loss given by $-q\lambda\nabla$ for state $s=-1$.

The new transition matrix becomes

$$\begin{aligned}
\mathbf{M}_\lambda &= \mathcal{E}_t \begin{bmatrix} p'_{12}+p'_{11} & p'_{10} & 0 \\ p'_{01} & p'_{00} & p'_{0-1} \\ 0 & p'_{-10} & p'_{-1-1}+p'_{-1-2} \end{bmatrix} \qquad (17) \\
&= \begin{bmatrix} 1-[q'\{1-a(1)d\}]-q\lambda & q'\{1-a(1)d\}+q\lambda & 0 \\ q'\{1+b(0)\,d\} & 1-q'\{1+b(0)\,d\}-q'\{1-a(0)d\} & q'\{1-a(0)d\} \\ 0 & q'\{1+b(-1)\,d\}+q\lambda & 1-q'\{1+b(-1)\,d\}-q\lambda \end{bmatrix} \\
&= \mathbf{M}+\begin{bmatrix} q\lambda\{1-a(1)d\}-q\lambda & -q\lambda\{1-a(1)d\}+q\lambda & 0 \\ -q\lambda\{1+b(0)\,d\} & q\lambda\{1+b(0)\,d\}+q\lambda\{1-a(0)d\} & -q\lambda\{1-a(0)d\} \\ 0 & -q\lambda\{1+b(-1)\,d\}+q\lambda & q\lambda\{1+b(-1)\,d\}-q.\lambda \end{bmatrix}.
\end{aligned}$$

and the other matrices change to

$$\begin{aligned}
\mathbf{\Lambda}'_0 &= \beta \begin{bmatrix} \left[-\frac{S}{2}-b(1)\right]p'_{12}+a(1)(p'_{10}-q\lambda)-\nabla q\lambda \\ -b(0)p'_{01}+a(0)p'_{0-1} \\ -b(-1)(p_{-10}-q\lambda)-\nabla q\lambda+\left[a(-1)-\frac{S}{2}\right]p_{-1-2} \end{bmatrix} \qquad (18) \\
&= \beta \begin{bmatrix} \left[-\frac{S}{2}-b(1)\right]p'_{12}+a(1)(p'_{10}-q\lambda) \\ -b(0)p'_{01}+a(0)p'_{0-1} \\ -b(-1)(p'_{-10}-q\lambda)+\left[a(-1)-\frac{S}{2}\right]p'_{-1-2} \end{bmatrix} + \beta q\lambda \begin{bmatrix} -\nabla \\ 0 \\ -\nabla \end{bmatrix} \\
&= \mathbf{\Lambda}_0+\mathbf{\Lambda}_\lambda,
\end{aligned}$$

$$\mathbf{\Lambda}_x = \beta \begin{bmatrix} 1+r \\ 0 \\ -(1+r) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},$$

$$\mathbf{\Phi} =\beta\mathcal{E}_t \begin{bmatrix} \Delta x_{t+1}\left(p'_{12}+p'_{11}\right) \\ \Delta x_{t+1}\left(p'_{01}-p'_{0-1}\right) \\ -\Delta x_{t+1}\left(p'_{-1-1}+p'_{-1-2}\right) \end{bmatrix} = \beta \begin{bmatrix} -q'd\mathcal{E}_t\left(\Delta x_{t+1}\right)^2 \\ -2q'd\mathcal{E}_t\left(\Delta x_{t+1}\right)^2 \\ -q'd\mathcal{E}_t\left(\Delta x_{t+1}\right)^2 \end{bmatrix} = \beta q' d\epsilon^2 \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix}.$$

The concavity parameter $\nabla = V(0) - V(1)$ is again implicitly characterized by

$$\mathbf{V}(s) = \begin{bmatrix} V(1) \\ V(0) \\ V(-1) \end{bmatrix} = \begin{bmatrix} V-\nabla \\ V \\ V-\nabla \end{bmatrix} = \left(\mathbf{I}-\beta\mathbf{M}_\lambda\right)^{-1}\left(\mathbf{\Lambda}_0+\mathbf{\Lambda}_\lambda+\mathbf{\Phi}\right). \tag{19}$$

where $\mathbf{\Lambda}_0$ has is the same term as before (except q is replaced by q') and we define a new term

$$\mathbf{\Lambda}_\lambda = \beta q\lambda \begin{bmatrix} -\nabla \\ 0 \\ -\nabla \end{bmatrix},$$

which reduces the cash flows in the two states with either high or low inventory. At the same time these two states become less likely as sophisticated customers contribute to more balanced dealer inventories. This also means that dealers need to rebalance in the B2B market less often.

Solving for the new B2C Equilibrium

Again we have to solve a system of three equations given by

$$\left(\mathbf{I}-\beta\mathbf{M}_\lambda\right)\mathbf{V}(s)-\mathbf{\Lambda}_0-\mathbf{\Lambda}_\lambda-\mathbf{\Phi}=\mathbf{0}.$$

Expanding, this gives us the following three equations:

$$\begin{bmatrix} 2q\beta\left(-1+\lambda\right)-d[4V\left(-1+\beta\right)+q\beta\left(-1+\lambda\right)\left(S-2\nabla\right)-4\left(-1+\beta\right)\nabla+d^2q\beta\left(-1+\lambda\right)\left(S^2-16\sigma^2+4\nabla^2\right)] \\ 2q\beta\left(-1+\lambda\right)-4d\left(V\left(-1+\beta\right)+2q\beta\left(-1+\lambda\right)\nabla\right)-2d^2q\beta\left(-1+\lambda\right)\left(4\sigma^2-\nabla^2\right) \\ 2q\beta\left(-1+\lambda\right)-d[4V\left(-1+\beta\right)+q\beta\left(-1+\lambda\right)\left(S-2\nabla\right)-4\left(-1+\beta\right)\nabla+d^2q\beta\left(-1+\lambda\right)\left(S^2-16\sigma^2+4\nabla^2\right)] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The first and last of three equations are identical because of the symmetry of the problem; hence we have two (non-linear) equations in two unknowns, V and ∇. It is straightforward to use the first equation to solve for V in terms of ∇ as

$$V = \frac{16d\left(-1+\beta\right)\nabla+q\beta\left(-1+\lambda\right)\left[8+d\left(-4S+dS^2+8\nabla+4d\left(-4\sigma^2+\nabla^2\right)\right)\right]}{16d\left(-1+\beta\right)}.$$

Substituting this into the second equation and solving for ∇, we obtain two solutions

$$\nabla = \frac{4-4\beta+6q\beta-6q\beta\lambda\pm\frac{1}{4}R}{2dq\beta\left(1-\lambda\right)},$$

where we define an expression

$$R = \sqrt{64\left(-2+\beta\left(2+3q\left(-1+\lambda\right)\right)\right)^2+16dq^2\beta^2\left(-1+\lambda\right)^2\left(-4S+dS^2+16d\sigma^2\right)}$$

To see that only the negative root is feasible, first recall that the ask quote $a(1) = \frac{1}{2}\left(\frac{1}{d} - \nabla\right) > 0$ and therefore $\nabla < \frac{1}{d}$; but for the positive root we obtain instead

$$\frac{3}{d} + \frac{4(1-\beta) + \frac{1}{4}R}{2dq\beta\,(1-\lambda)} > \frac{1}{d}.$$

Hence, the correct solution for the B2C schedule is provided by the negative root.

The properties of the B2C curve in (∇, S) space (Figure 3) do not dramatically change by the inclusion of sophisticated traders; as before we have

$$\frac{\partial\nabla}{\partial S} = \frac{q\beta\,(-4+2dS)\,(-1+\lambda)}{R} > 0$$

The sign follows because $\frac{S}{2} < \frac{1}{d}$ and $0 \leq \lambda < 1$. As $\lambda \to 1$, the slope of the of the B2C curve in Figure 2 tends towards zero. It is tedious but straightforward to show that $\frac{\partial^2\nabla}{\partial S^2}$ is always negative under our assumptions (Mathematica file available on request). The intercept of the B2C with the horizontal axis is independent of λ and given by

$$\frac{S}{2} = \frac{1}{d} - \frac{\sqrt{1-4d^2\sigma^2}}{d}.$$

It is obvious from inspection of the expression for ∇ (recalling that we are using the negative root) that $\frac{\partial\nabla}{\partial\sigma^2} < 0$. Hence increased volatility shifts the B2C curve downwards. Finally, ∇ is decreasing in the fraction of sophisticated traders:

$$\frac{\partial\nabla}{\partial\lambda} = -\frac{2\nabla\left(\frac{1-\beta}{1-\lambda}\right)}{R} < 0$$

whenever $\nabla > 0$.